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Natural Relations in the Standard Model

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Abstract

We establish the potential existence of *natural relations* between the Cabibbo angle and the quark mass ratios, in a Standard Model with one Higgs doublet and two quark generations. The argument is based on the calculation of the divergent one-loop radiative corrections to the quark mass matrices.

One of the outstanding questions raised by the Standard Model (SM) of elementary particles concerns the set of free parameters it seems to involve. Many of those parameters find their origin in the Yukawa sector of the theory: six quark masses, three mixing angles and one phase, not to mention the leptons. Furthermore, their empirical values show a strongly hierarchical pattern. Since the early times of its discovery, one has tried to complete the SM in order to reduce the size of its set of free parameters and hence propose an explanation for this observed hierarchy. The simplest way to reach this goal is to look for potential relations between apparently free parameters. But one has to make sure that those relations are preserved by the renormalization; they must be *natural* [1, 2]. Until now, most attempts consisted in enlarging the symmetry group of the SM by adding a horizontal component to it [3, 4, 5, 6]. The horizontal symmetry imposes constraints on the structure of the Yukawa couplings. After spontaneous breakdown of the symmetry, the fermion mass matrices that are generated still bear the stamp of those constraints and through bidiagonalization, they give rise to relations between mass ratios and mixing angles. Such an implementation guarantees that the relations survive to renormalization; they are called *natural*. However one soon realized [7, 8, 9, 10] that it could not be achieved

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without extending the particle content – by considering models with more than one Higgs doublet, thereby increasing the number of couplings...

We consider the SM in its minimal realization, namely built up with one single Higgs doublet. We ignore the leptons and concentrate on the quark sector. The approach we put forward here is based on the calculation of the divergent one-loop radiative corrections to the quark mass matrices, which exclusively involves self-energy and tadpole diagrams. More precisely, since we are interested in natural relations between up-type quark mass ratios and down-type quark mass ratios on the one hand, and mixing angles on the other hand, we will solely compute the divergent one-loop radiative corrections to those specific parameters. This considerably simplifies our task. One indeed notices that neither QED nor QCD, which are flavour-blind, will bring in divergent contributions that would affect the mixing angle or the mass ratios. The same argument holds for the diagrams involving the transverse polarizations of the Z^0 and of the $W^\pm$ vector bosons, as well as the tadpoles. To persuade oneself, it is worth checking that the contribution of the latter diagrams to the renormalization of the quark mass ratios reads

$$\frac{m_u}{m_c} \longmapsto \frac{(1 + C_\gamma + C_G + C_{Z^0} + C_{W^\pm} + C_T)m_u}{(1 + C_\gamma + C_G + C_{Z^0} + C_{W^\pm} + C_T)m_c} = \frac{m_u}{m_c}$$

where C_γ, C_G, C_{Z^0} and $C_{W^\pm}$ respectively originate from the interventions of the photon, the gluons, the transverse Z^0 and the transverse $W^\pm$ in the up-type quark self-energies, while C_T originates from the tadpole diagrams. Those C's are identical for the mass renormalization of any up-type quark – as far as the divergent part is concerned. The reasoning can be applied to the down-type quarks. For the mixing angles, the proof is even more direct since none of those diagrams leads to divergent non-diagonal correction to the tree-level diagonal mass matrices. In other words, the only diagrams one has to consider are the quark self-energies due to the exchange of the scalars (Higgs and would-be-Goldstone bosons) – for the complete list of the relevant divergent diagrams, see figure 1. This is not astonishing since the scalars are the only fields that know about the difference between the fermion families. We are looking for a special structure inside the Yukawa couplings (or the mass matrices), hence the fact that the Yukawa sector is the only one responsible for the naturalness of this structure will come as no surprise.

We look into a two-quark-generation SM. Once we have the expression of the divergent one-loop radiative corrections to the Cabibbo angle and to the two mass ratios, we suppose the existence of a tree-level natural relation between them. Since it is natural, this putative relation must hold at the one-loop level up to some finite corrections. This gives us a severe criterion to constrain the shape of any natural relation. We show that there are indeed potential one-loop natural relations in a SM with one Higgs doublet and two quark generations, contrary to what has been thought so far. We conclude that those relations cannot originate from any additionnal horizontal symmetry. We finally state our a result in a different way and claim there is one apparently free parameter, in a SM with one Higgs doublet and two quark generations, to which the one-loop radiative corrections are finite. In other words, one apparently free parameter whose presence in the SM seems not to be necessary to render it renormalizable; or else, one apparently free parameter whose one-loop β-function vanishes. We call it a *stable* parameter.

Let us briefly recall the input we need. We start from a two-fermion-generation SM; the scalar sector includes one single Higgs doublet. The hadronic Yukawa sector, which we are interested in, contains three useful free parameters: the Cabibbo angle θ [11] and the two mass ratios $r_u = m_u/m_c$ and $r_d = m_d/m_s$. The approach is the following: we assume the existence of a natural relation between the Cabibbo angle and the mass ratios; we then make use of the definition of a natural relation to establish its general form. Namely, assuming the existence of a natural relation automatically determines that relation.

A general natural relation between the Cabibbo angle and the mass ratios reads:

$$F(\theta) = G(r_u, r_d) \tag{1}$$

so that

$$F(\theta + \delta\,\theta) = G(r_u + \delta\,r_u, r_d + \delta\,r_d) \tag{2}$$

where $\delta\,\theta$, $\delta\,r_u$ and $\delta\,r_d$ are the divergent parts of the radiative corrections to θ, r_u and r_d respectively. This last expression is the mathematical translation of naturalness. To be natural, (1) cannot be broken by infinite radiative corrections. That is, (2) must be verified. Thus

$$F_{|\theta}\,\delta\,\theta = G_{|r_u}\,\delta\,r_u + G_{|r_d}\,\delta\,r_d$$

where $f_{|x}$ denotes the (partial) derivative of f with respect to x. Computing the one-loop radiative corrections to the Yukawa couplings (see Appendix), one obtains:

$$\delta\,\theta = \epsilon \left[\frac{1+r_u^2}{1-r_u^2}(m_d^2 - m_s^2) + \frac{1+r_d^2}{1-r_d^2}(m_u^2 - m_c^2) \right] \sin\theta\cos\theta$$

$$\delta\, r_u = \epsilon\; r_u\; [(m_d^2 - m_s^2)\cos 2\theta - (m_u^2 - m_c^2)] \tag{3}$$

$$\delta\, r_d = \epsilon\; r_d\; [(m_u^2 - m_c^2)\cos 2\theta - (m_d^2 - m_s^2)]$$

with $\epsilon = \frac{3}{4}\frac{2}{v^2}\left(\frac{1}{4\pi^2}\ln\frac{\Lambda^2}{\mu^2}\right)$, Λ being a cut-off, μ an arbitrary energy scale and v the vacuum expectation value of the scalar field. Let us bear in mind that we do not care about the finite part of the corrections. Introducing those expressions into equation (2) splits it into two independent equations (since we are not interested in considering relations involving mass ratios different from r_u and r_d):

$$F_{|\theta}\sin\theta\cos\theta\,\frac{1+r_u^2}{1-r_u^2} = G_{|r_u}\, r_u\cos 2\theta - G_{|r_d}\, r_d \tag{4}$$

$$F_{|\theta}\sin\theta\cos\theta\,\frac{1+r_d^2}{1-r_d^2} = G_{|r_d}\, r_d\cos 2\theta - G_{|r_u}\, r_u \tag{5}$$

which turn out to be compatible if and only if

$$\cos 2\theta = \frac{\frac{1-r_u^2}{1+r_u^2}\, r_d\, G_{|r_d} - \frac{1-r_d^2}{1+r_d^2}\, r_u\, G_{|r_u}}{\frac{1-r_u^2}{1+r_u^2}\, r_u\, G_{|r_u} - \frac{1-r_d^2}{1+r_d^2}\, r_d\, G_{|r_d}} \tag{6}$$

Now this *must* be the natural relation (1) whose existence has been supposed, i.e.[1]

$$F(\theta) = \cos 2\theta \tag{7}$$

and

$$G(r_u, r_d) = \frac{\frac{1-r_u^2}{1+r_u^2}\, r_d\, G_{|r_d} - \frac{1-r_d^2}{1+r_d^2}\, r_u\, G_{|r_u}}{\frac{1-r_u^2}{1+r_u^2}\, r_u\, G_{|r_u} - \frac{1-r_d^2}{1+r_d^2}\, r_d\, G_{|r_d}} \tag{8}$$

[1] One checks that the arbitrary character of those identifications will not show itself in the expected solution. To prove it, we imagine (7) would rather read $f(F(\theta)) = \cos 2\theta$. One should then replace G by $f(G)$ in the left-hand side of (8). But the right-hand side of it is invariant under $G \mapsto f(G)$. Namely, one can solve (8) with respect to the variable $f(G)$ which we finally identify to $f(F(\theta)) = \cos 2\theta$.

Exploiting the remaining information in (4) and (5), and using (6) and (7), yields

$$\begin{cases} \frac{1+r_u^2}{1-r_u^2} G - r_u G_{|r_u} + \frac{1+r_d^2}{1-r_d^2} = 0 \\ \frac{1+r_d^2}{1-r_d^2} G - r_d G_{|r_d} + \frac{1+r_u^2}{1-r_u^2} = 0 \end{cases}$$

This system – from which one obviously recovers equation (8) – is integrable, and the general solution reads

$$G(r_u, r_d) = \frac{-(1+r_u^2)(1+r_d^2) + 2\lambda\, r_u r_d}{(1-r_u^2)(1-r_d^2)}$$

where λ is the integration constant. One concludes that, if a relation of the kind suggested in (1) exists, it necessarily belongs to the following class:

$$\cos 2\theta = \frac{-(m_u^2 + m_c^2)(m_d^2 + m_s^2) + 2\lambda\, m_u m_c m_d m_s}{(m_u^2 - m_c^2)(m_s^2 - m_d^2)} \tag{9}$$

We have found an infinite number of potential one-loop natural relations inside the Yukawa sector. This set is parametrized by a dimensionless constant λ. One has no further theoretical argument to constrain the value of λ; and one cannot evade the difficulty by asking one of the quark masses to vanish, since it would lead to a cosine smaller than minus one[2]. Now, one has to select the unique viable natural relation by making λ fit the data, i.e.

$$\lambda = \frac{(m_u^2 + m_c^2)(m_d^2 + m_s^2) + (m_u^2 - m_c^2)(m_d^2 - m_s^2)\cos 2\theta}{2 m_u m_c m_d m_s} \tag{10}$$

This last step gives the impression one goes round in a circle; it does of course not increase the predictive power of the reasoning, but it allows us to shed a new light on the result. One might indeed think of λ as a physical quantity (since it is built from physical ones) which is *not* renormalized, i.e. which is not useful for absorbing divergences. Or else, the one-loop λ differs from the tree-level one by a finite quantity. In other words, the Sandard Model with two quark generations and one Higgs doublet seems to be vast enough to allow one of its apparently free parameters not to absorb any divergent

[2]Consequently we may already conclude that it is not possible to naturally set the mass of one single quark to zero, together with the requirement of the existence of a natural relation between the Cabibbo angle and the quark mass ratios. In this context, by excluding $m_u = 0$ our result rules out the natural vanishing of the QCD vacuum parameter θ_S.

one-loop radiative correction (we show in the Appendix that $\delta\lambda = 0$ as expected by construction,). This means that the one-loop β-function of λ vanishes, i.e. at the one-loop level, λ does not run; λ is *stable*.

Stating that several natural relations potentially exist in a one-Higgs-doublet model apparently contradicts previous results obtained in the context of family symmetries [7, 8, 9, 10]. But since we do not appeal to such kind of symmetries, we do not expect our result to respect the conclusions derived in their context. Namely, the potential natural relations (9) cannot be associated with the presence of any extra horizontal symmetry[3]. One should examine the validity of the results at the n-loop level; then look for the reason of the existence of a *stable* parameter inside the SM; and finally look for some possible "determination principle" of it outside or beyond the SM.

The analysis we have conducted here can be applied to the leptons, provided that the neutrinos are massive, their mass being of the Dirac type exclusively. The result, since it depends on the sole structure of the Yukawa sector, is strictly identical.

The extension of the present calculation to a three-quark-generation SM seems to be doomed to failure because of the complexity of the one-loop radiative corrections to the Cabibbo-Kobayashi-Maskawa parameters [11, 12]. Those corrections are indeed too cumbersome to be manipulated and introduced into a solvable partial differential equations system. We will however expound, in a forthcoming paper, an alternative approach to derive similar results in a n-quark-generation SM.

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Appendix

[3]According to [8], in a one-Higgs-doublet model, the Cabibbo angle which is determined by a family symmetry can only take the values 0 or $\pi/2$, yielding $\cos 2\theta = 1$ or $\cos 2\theta = -1$ respectively. In our approach, (10) then reads $\lambda = (m_u^2 m_d^2 + m_c^2 m_s^2)/(m_u m_c m_d m_s)$ or $\lambda = (m_c^2 m_d^2 + m_u^2 m_s^2)/(m_u m_c m_d m_s)$, and there still is an opportunity for relating the mass ratios among each other. However, again, this potential natural relation is not to be regarded as the consequence of the presence of any family symmetry.

A Self-energies in the Yukawa sector

After spontaneous breakdown of the symmetry, the Yukawa sector Lagrangian for quarks reads:

$$
\begin{array}{rcccccccc}
\mathcal{L}_Y & = & \bar{u}_L\Gamma_d d_R\phi^+ & + & \bar{d}_L\Gamma_d d_R\phi^0 & + & \bar{d}_R\Gamma_d^\dagger u_L\phi^- & + & \bar{d}_R\Gamma_d^\dagger d_L\phi^{0\star} \\
 & + & \bar{u}_L\Gamma_u u_R\phi^{0\star} & - & \bar{d}_L\Gamma_u u_R\phi^- & + & \bar{u}_R\Gamma_d^\dagger u_L\phi^0 & - & \bar{u}_R\Gamma_u^\dagger d_L\phi^+ \\
 & + & \bar{d}_L M_d d_R & + & \bar{u}_L M_u u_R & + & \bar{d}_R M_d^\dagger d_L & + & \bar{u}_R M_d^\dagger u_L
\end{array}
$$

The Lagrangian fields and parameters are renormalized:

$$
\begin{array}{rcccc}
u_{L,R} & \longmapsto & u'_{L,R} & = & (Z^u_{L,R})^{-\frac{1}{2}} u_{L,R} \\
d_{L,R} & \longmapsto & d'_{L,R} & = & (Z^d_{L,R})^{-\frac{1}{2}} d_{L,R} \\
M_u & \longmapsto & M'_u & = & (Z^u_L)^{\frac{1}{2}} (Z_{M_u})^{-1} M_u (Z^u_R)^{\frac{1}{2}} \\
M_u & \longmapsto & M'_d & = & (Z^d_L)^{\frac{1}{2}} (Z_{M_d})^{-1} M_d (Z^d_R)^{\frac{1}{2}}
\end{array}
$$

The one-loop calculation of the fermion self-energies leads to (the first term corresponding to the neutral current intervention ; the second, to the charged current intervention – see figure 1):

$$
\begin{array}{cc}
Z^u_L = 1 - \frac{\epsilon}{2}[\Gamma_u\Gamma_u^\dagger + \Gamma_d\Gamma_d^\dagger] & Z^d_L = 1 - \frac{\epsilon}{2}[\Gamma_d\Gamma_d^\dagger + \Gamma_u\Gamma_u^\dagger] \\
Z^u_R = 1 - \frac{\epsilon}{2}[\Gamma_u\Gamma_u^\dagger + \Gamma_u\Gamma_u^\dagger] & Z^d_R = 1 - \frac{\epsilon}{2}[\Gamma_d\Gamma_d^\dagger + \Gamma_d\Gamma_d^\dagger] \\
Z_{M_u} = 1 - \epsilon[0 + \Gamma_d\Gamma_d^\dagger] & Z_{M_d} = 1 - \epsilon[0 + \Gamma_u\Gamma_u^\dagger]
\end{array}
\tag{11}
$$

with $\epsilon = \left(\frac{1}{4\pi^2}\ln\frac{\Lambda^2}{\mu^2}\right)$ where Λ is a cut-off and μ an arbitrary energy scale (one checks that $Z^u_L = Z^d_L$ as expected). Those results are true only up to a term proportional to the identity in the flavour space, which would take into account the electromagnetic, weak transversal and strong contributions, as well as the tadpole ones. But since this term would factor out in the final result, which is supposed to involve exclusively mixing parameters and *up*- or *down*-type mass ratios, we chose not to write it down. The finite parts of the diagrams are omitted.

From (11), one derives the mass matrices corrections in the weak base

$$
\begin{aligned}
M'_u &= M_u + \epsilon\,[M_d M_d^\dagger M_u - M_u M_u^\dagger M_u] \\
M'_d &= M_d + \epsilon\,[M_u M_u^\dagger M_d - M_d M_d^\dagger M_d]
\end{aligned}
$$

and in the physical base

$$
\begin{aligned}
U_L^\dagger M'_u U_R &= D_u + \epsilon\,[K D_d^2 K^\dagger D_u - D_u^3] \\
V_L^\dagger M'_d V_R &= D_d + \epsilon\,[K^\dagger D_u^2 K D_d - D_d^3]
\end{aligned}
$$

where we have absorbed a $\frac{3}{4}\frac{2}{v^2}$ factor in ϵ, v being the scalar VEV, and where D_u and D_d are the tree-level diagonal mass matrices while K is the tree-level Cabibbo-Kobayashi-Maskawa matrix. Let us repeat that those expressions do not include the finite parts of the radiative correction, and that they account for the sole (neutral and charged) scalar exchanges in the fermion self-energies.

One can rewrite those last expressions as follows

$$
\begin{aligned}
M''_u &= D_u + \epsilon_u \\
M''_d &= D_d + \epsilon_d
\end{aligned}
$$

and proceed to the diagonalization of M''_u and M''_d, i.e.

$$
\begin{aligned}
U'^{\dagger}_L M''_u U'_R &= D'_u \\
V'^{\dagger}_L M''_d V'_R &= D'_d
\end{aligned}
$$

where

$$
D'_u = \begin{pmatrix} m_u + \epsilon_{u\,11} & \\ & m_c + \epsilon_{u\,22} \end{pmatrix} \qquad U'_L = \begin{pmatrix} 1 & \theta_u \\ -\theta_u & 1 \end{pmatrix}
$$

$$
D'_d = \begin{pmatrix} m_d + \epsilon_{d\,11} & \\ & m_s + \epsilon_{d\,22} \end{pmatrix} \qquad V'_L = \begin{pmatrix} 1 & \theta_d \\ -\theta_d & 1 \end{pmatrix}
$$

with

$$
\theta_u = \frac{m_u \epsilon_{u\,21} + m_c \epsilon_{u\,12}}{m_c^2 - m_u^2} \qquad \text{and} \qquad \theta_d = \frac{m_d \epsilon_{u\,21} + m_s \epsilon_{u\,12}}{m_s^2 - m_d^2}
$$

The one-loop mixing matrix is defined by

$$
K' = U'^{\dagger}_L K V'_L
$$

so that the one-loop mixing angle reads

$$
\theta' = \theta - \theta_u + \theta_d
$$

Inserting the value of ϵ_u and ϵ_d in θ', D'_u and D'_d, leads to the equations (3).

B Divergent radiative corrections to λ

$$
\begin{aligned}
\delta\lambda &= \lambda_{,\theta}\delta\theta + \lambda_{,r_u}\delta r_u + \lambda_{,r_d}\delta r_d \\
&= -2\sin 2\theta \frac{(1-r_u^2)(1-r_d^2)}{2r_u r_d}\delta\theta \\
&\quad + \frac{-r_d(1-r_u^2)(1+r_d^2) - r_d(1+r_u^2)(1-r_d^2)\cos 2\theta}{2r_u^2 r_d^2}\delta r_u \\
&\quad + \frac{-r_u(1+r_u^2)(1-r_d^2) - r_u(1-r_u^2)(1+r_d^2)\cos 2\theta}{2r_u^2 r_d^2}\delta r_d \\
&= -\epsilon \sin^2 2\theta \left[\frac{(1+r_u^2)(1-r_d^2)}{2r_u r_d}(m_d^2 - m_s^2) + \frac{(1-r_u^2)(1+r_d^2)}{2r_u r_d}(m_u^2 - m_c^2)\right] \\
&\quad + \epsilon \sin^2 2\theta \frac{(1+r_u^2)(1-r_d^2)}{2r_u r_d}(m_d^2 - m_s^2) \\
&\quad + \epsilon \sin^2 2\theta \frac{(1-r_u^2)(1+r_d^2)}{2r_u r_d}(m_u^2 - m_c^2) \\
&= 0
\end{aligned}
$$

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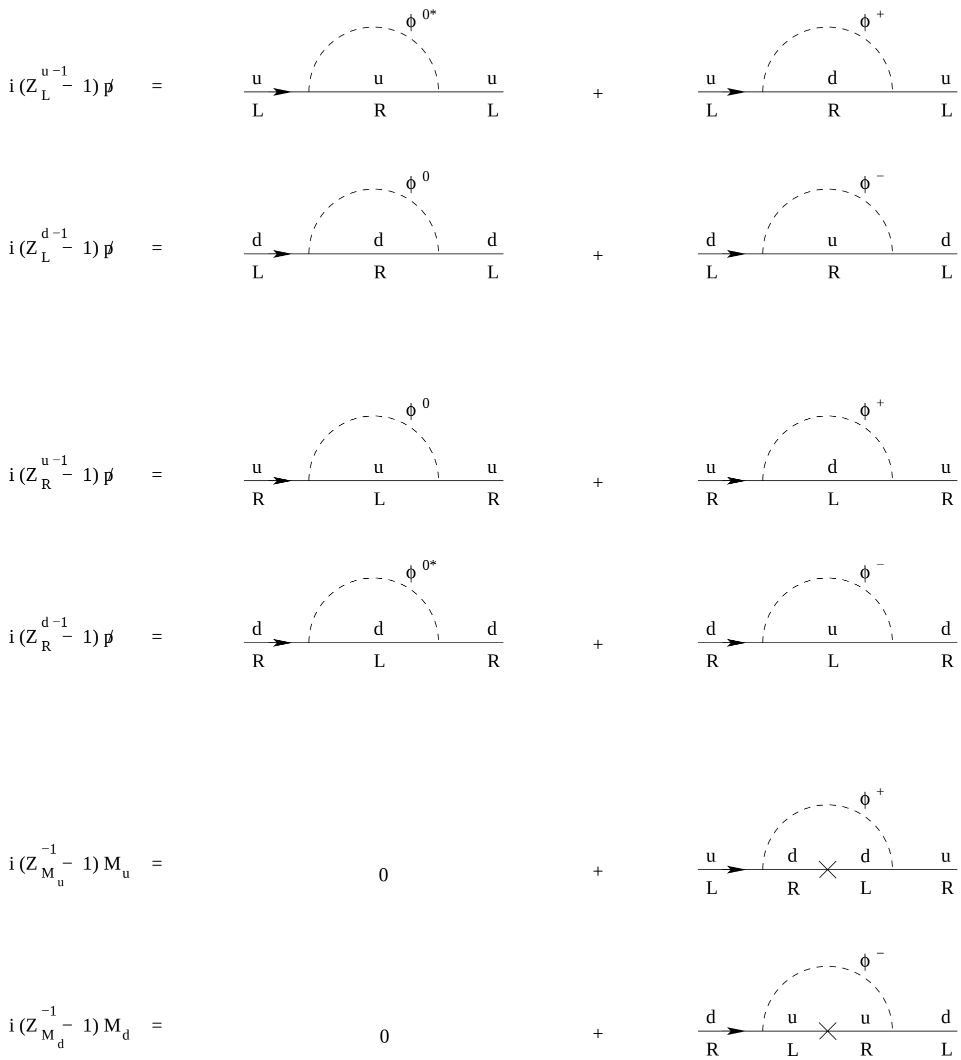

Fig. 1: relevant divergent diagrams involved in the calculation of the renormalization constants.